LECTURE 8

INTERPOLATION USING CHEBYSHEV ROOTS

- For both Lagrange and Newton interpolation through $N+1$ data points (N^{th} degree polynomial which fits through all $N+1$ data points).

$$e(x) = \frac{(x-x_o)(x-x_1)(x-x_2)\ldots(x-x_N)}{(N+1)!} f^{(N+1)}(\xi) \qquad x_o < \xi < x_N$$

$$\Rightarrow$$

$$e(x) = \frac{1}{(N+1)!}\prod_{i=0}^{N}(x-x_i)\; f^{(N+1)}(\xi) \qquad x_o < \xi < x_N$$

- Typically over a small interval, $f^{(N+1)}(\xi)$ won't change dramatically (although strictly speaking ξ ***does*** depend on x)

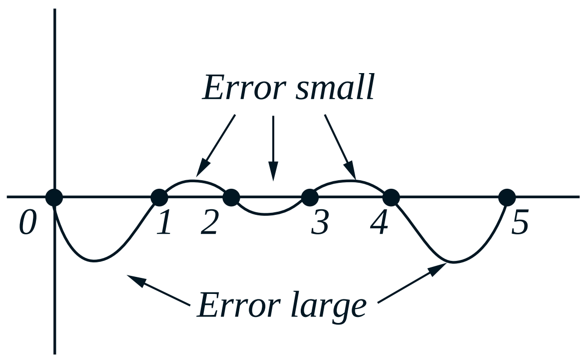
$$f^{(N+1)}(\xi) \cong f^{(N+1)}(x_m) \qquad \text{where} \quad x_m = \frac{x_o + x_N}{2}$$

- We can not control the $f^{(N+1)}(\xi)$ portion of $e(x)$ since we don't know f and we can't specify ξ.

- The error is for the most part controlled by $\prod_{i=1}^{N}(x - x_i)$

- $\prod_{i=0}^{N}(x - x_i)$ is small in the center of the interval $[x_o, x_N]$, but large within the end zones.

  - For example for *N*=*5*, we examine a plot of $\prod_{i=0}^{5}(x - x_i)$:



- ***Our objective is to minimize*** $\boldsymbol{e(x)}$ ***by minimizing*** $\boldsymbol{\prod_{i=1}^{N}(x - x_i)}$ ***by selecting a "special" set of non-equispaced interpolating points***

Chebyshev Polynomials

- Consider only the interval $-1 \le x \le 1$. (We will generalize to any interval at a later point.)
- The Chebyshev polynomial is defined as:

$$\boldsymbol{T_j(x) \equiv \cos(j\cos^{-1}x) \quad \text{on} \quad -1 \le x \le 1 \qquad j = 0, 1, \ldots}$$

- Note that the $\cos^{-1}x$ term restricts the range since $|x| \ge 1$ is ***not*** defined.

- Zero[th] Degree Chebyshev Polynomial:

$$T_o(x) = \cos(0 \cdot \cos^{-1}x) \qquad \Rightarrow$$

$$T_o(x) = 1$$

- First Degree Chebyshev Polynomial:

$$T_1(x) = \cos(1\cos^{-1}x) \qquad \Rightarrow$$

$$T_1(x) = x$$

- Second Degree Chebyshev Polynomial:

$$T_2(x) = \cos(2\cos^{-1}x)$$

  - From the CRC Math Handbook: Double Angle Relation $\cos 2\alpha = 2\cos^2\alpha - 1$

$$T_2(x) = 2\cos^2(\cos^{-1}x) - 1 \qquad \Rightarrow$$

$$T_2(x) = 2[\cos(\cos^{-1}x)][\cos(\cos^{-1}x)] - 1 \qquad \Rightarrow$$

$$T_2(x) = 2x^2 - 1$$

- Third Degree Chebyshev Polynomial:

$$T_3 = \cos(3\cos^{-1}x)$$

  - From the CRC Math Handbook: Multiple Angle Relation $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$$T_3(x) = 4[\cos(\cos^{-1}x)]^3 - 3\cos(\cos^{-1}x) \qquad \Rightarrow$$

$$T_3(x) = 4x^3 - 3x$$

- In general recursive relationship can be developed

$$T_j(x) = 2xT_{j-1}(x) - T_{j-2}(x)$$

- Chebyshev polynomials can be normalized

$$\boldsymbol{\psi_j(x) \equiv \frac{T_j(x)}{2^{j-1}}}$$

  - We note that for $\psi_j(x)$, the coefficient of the highest degree polynomial term equals unity. For example:

$$\psi_3(x) = \frac{T_3(x)}{2^2} = x^3 - \frac{3}{4}x$$

Properties of Chebyshev Polynomials

- Cosine of ***any*** argument will always range between $-1 \le \cos\alpha \le 1$, therefore:

$$T_j(x) \equiv \cos(j\cos^{-1}x) \quad \text{ranges between} \quad -1 \le T_j(x) \le 1$$

- The normalized Chebyshev polynomial:

$$\psi_j(x) \equiv \frac{\cos(j\cos^{-1}(x))}{2^{j-1}} \quad \textbf{ranges between} \quad \frac{-1}{2^{j-1}} \le \psi_j(x) \le \frac{1}{2^{j-1}}$$

- Let's examine the roots of the Normalized Chebyshev Polynomial, $\psi_{N+1}(x)$.

$$\psi_{N+1}(x^c) = 0 \quad \Rightarrow$$

$$T_{N+1}(x^c) = 0 \quad \Rightarrow$$

$$\cos[(N+1)\cos^{-1}(x^c)] = 0 \quad \Rightarrow$$

$$(N+1)\cos^{-1}(x^c) = \cos^{-1}(0)$$

- Since $\cos^{-1}(0) = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$$(N+1)\cos^{-1} x_{n-1}^{c} = \left(N+1+\frac{1}{2}-n\right)\pi \qquad n = 1, 2, \ldots, N+1$$

where the n values are selected such that all roots falling within the range $-1 \le x \le 1$ are defined. Extending the range for n will only lead to repeated roots. Thus

$$\boldsymbol{x_{n-1}^{c} = \cos\left[\frac{\left(N+1+\frac{1}{2}-n\right)\pi}{N+1}\right] \qquad n = 1, 2, \ldots, N+1}$$

- We note that since $x_o^c, x_1^c, x_2^c, \ldots$ are the roots of $\psi^{N+1}(x)$ we may write the following

$$\psi_{N+1}(x) = (x - x_o^c)(x - x_1^c)\ldots(x - x_N^c) \quad \Rightarrow$$

$$\boldsymbol{\psi_{N+1}(x) = \prod_{i=0}^{N}(x - x_i^c)}$$

Example

- Find the Chebyshev roots for the $N+1 = 3$ degree Chebyshev polynomial

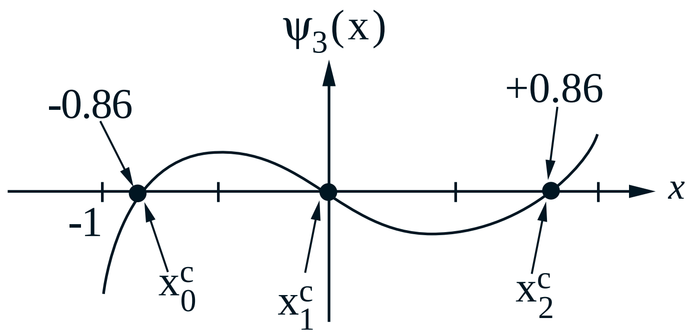
$$\psi_3(x) = x^3 - \frac{3}{4}x$$

- The roots are computed as:

$$x_o^c = \cos\left[\frac{3 + \frac{1}{2} - 1}{3} \cdot \pi\right] = \cos\left(\frac{5}{6}\pi\right) = -0.866025$$

$$x_1^c = \cos\left[\frac{3 + \frac{1}{2} - 2}{3} \cdot \pi\right] = \cos\frac{\pi}{2} = 0$$

$$x_2^c = \cos\left[\frac{3 + \frac{1}{2} - 3}{3} \cdot \pi\right] = 0.866025$$



- We note that the product of $(x-x_0^c)(x-x_1^c)(x-x_2^c)$ in fact equals the third degree Chebyshev polynomial, $\psi_3(x)$. Thus

$$(x+0.866025)(x-0)(x-0.866025) = x(x^2-0.75) = x^3-\frac{3}{4}x = \psi_3(x)$$

- ***We note that this type of polynomial product term (with the roots defined as the interpolating points) appeared in the error formula for Lagrange interpolation***

Application of Chebyshev Roots as Interpolation Points

- In general if $x_o^c, x_1^c, x_2^c, \ldots$ are the roots of $\psi^{N+1}(x)$, then

$$\psi_{N+1}(x) = \prod_{i=0}^{N}(x - x_i^c)$$

- For Lagrange interpolation through $N+1$ data points, the error function is expressed as:

$$e(x) = \frac{(x-x_o)(x-x_1)(x-x_2)\ldots(x-x_N)}{(N+1)!} f^{(N+1)}(\xi)$$

- ***Thus if we select the roots of the $N+1$ degree Chebyshev polynomial as our interpolation (or data) points for Lagrange Interpolation (or any N degree polynomial interpolation scheme with variably spaced data points)***

$$e^c(x) = \frac{1}{(N+1)!}\psi_{N+1}(x) f^{(N+1)}(\xi)$$

- Notes:
  - This error estimate is only good for the case where the roots of the Chebyshev polynomial are used as interpolation points and the interval is $-1 \le x \le 1$
  - Since the magnitude of $\psi_{N+1}(x)$ (the normalized Chebyshev polynomial) is minimized to

$$-\frac{1}{2^N} \le \psi_{N+1}(x) \le \frac{1}{2^N}$$

    we have effectively minimized the maximum error $e(x)$ over the interval (as far as we can)!
  - The distribution of error is now more even on the interval.
  - We haven't entirely minimized $e(x)$ since ξ depends on x and we can't do much about this.

Example

- Develop an interpolation formula over the range $-1 \le x \le 1$ with 3 data points which minimizes the maximum error over the interval. Estimate the maximum error over the interval.

- The roots of the Chebyshev polynomial with $N + 1 = 3$ are:

$$x_o^c = -0.866, \quad x_1^c = 0, \quad x_2^c = 0.866$$

- The Lagrange interpolating function is:

$$g(x) = f_o \frac{(x - x_1^c)(x - x_2^c)}{(x_o^c - x_1^c)(x_o^c - x_2^c)} + f_1 \frac{(x - x_o^c)(x - x_2^c)}{(x_1^c - x_o^c)(x_1^c - x_2^c)} + f_2 \frac{(x - x_o^c)(x - x_1^c)}{(x_2^c - x_o^c)(x_2^c - x_1^c)}$$

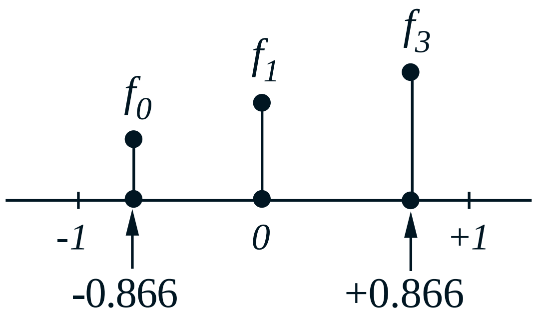
- Note that the functional values f_o, f_1 and f_2 are now evaluated at the Chebyshev roots

$$f_o = f(x_o^c), \quad f_1 = f(x_1^c) \text{ and } f_2 = f(x_2^c)$$

- Substituting in values for x_o^c, x_1^c and x_2^c

$$g(x) = f_o\frac{(x-0)(x-0.866)}{(-0.866-0)+(-0.866-0.866)} + f_1\frac{(x+0.866)(x-0.866)}{(0+0.866)(0-0.866)}$$

$$+ f_2\frac{(x+0.866)(x-0)}{(0.866+0.866)(0.866-0)}$$



- Note that for $[-1, -0.866]$ and $[0.866, 1]$ we are strictly speaking extrapolating, not interpolating.
  - However since we have carefully placed the points, we will not incur excessive errors in these extrapolated ranges.

- The maximum error over the interval may be estimated as ($N + 1 = 3$):

$$e^c(x) = \frac{1}{3!}\psi_3(x) f^{(3)}(\xi), \qquad -1 \le \xi \le 1 \qquad \Rightarrow$$

$$e^c(x) \cong \frac{1}{3!}\psi_3(x) f^{(3)}(x_m), \qquad x_m = 0 \qquad \Rightarrow$$

$$max|e^c(x)| = \frac{1}{6} max(\psi_3(x)) f^3(x_m)$$

- However we noted that

$$-\frac{1}{2^N} \le \psi_{N+1}(x) \le \frac{1}{2^N}$$

- Thus over the interval

$$max|e^c(x)| = \frac{1}{6} \cdot \frac{1}{2^2} \; f^{(3)}(x_m) \qquad \Rightarrow$$

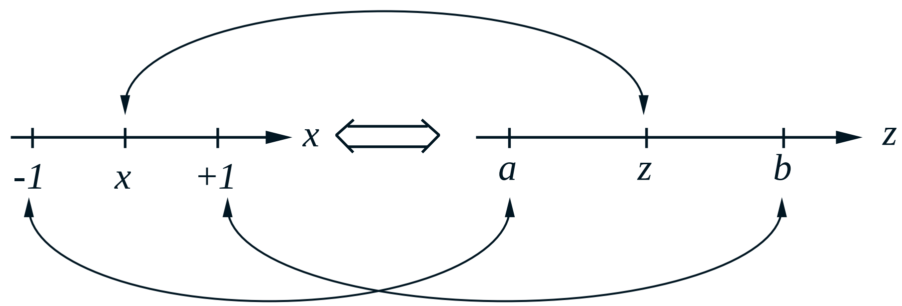
$$\boldsymbol{max|e^c(x)| = \frac{1}{24} f^{(3)}(x_m)}$$

- Note that $f^{(3)}(x_m)$ can be estimated using a forward/backward difference formula

Generalization of the Interpolation Interval

- So far we considered $[-1, +1]$
- We can map $[-1, +1]$ to the range of interest $[a, b]$ with the transformation

$$x = \frac{2z - a - b}{b - a} \quad \Leftrightarrow \quad z = \frac{(b-a)x + a + b}{2}$$



- Use this transformation and substitute for x in all formulae. The Chebyshev roots or interpolation points (nodes) become:

$$z^c_{n-1} = \frac{1}{2}\left[(b-a)\cos\left(\frac{N + 1 + \frac{1}{2} - n}{N+1}\pi\right) + a + b\right], \quad n = 1, 2, \ldots, N+1$$